

Summary of PhD thesis

## *Matrices of exceptional modules and matrix factorizations of vector bundles*

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The dissertation deals with two issues of the theory of representation of algebras. The first problem is to determine all possible entries in the matrices of representations attached to exceptional modules over wild canonical algebras. The second problem is to determine the matrix factorizations for the triangle singularities  $f = x^a + y^b + z^c$  of domestic type, that is, we assume that  $(a, b, c)$  are integers greater than or equal to two, satisfying  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$ . The starting point of these issues is category of coherent sheaves  $\text{coh}(\mathbb{X})$  over a weighted projective line  $\mathbb{X}$  in the sense of Geigle-Lenzing. This category is derived equivalent to the category of right modules over the associated canonical algebra. Considerations in the dissertation are conducted over an algebraically closed fields.

### **Representations of exceptional modules**

The canonical algebras were introduced by C. M. Ringel and they are one of the most important classes of algebra. Concerning the complexity of the module category there are three types of canonical algebras: domestic, tubular and wild. In the case of domestic type D. Kussin and H. Meltzer have determined matrices for all indecomposable representations. In the case of tubular canonical algebras, H. Meltzer examined possible entries of the matrices describing the exceptional modules using universal extensions in the sense of Bongartz.

The purpose of the thesis is to study the possible coefficients in the matrices of exceptional modules over wild canonical algebras. The approach presented in the dissertation generalizes the considerations of C.M. Ringel in the case of path algebras without relations and of H. Meltzer for tubular canonical algebras.

First, Schofield's induction is adopted in the category of coherent sheaves over a weighted projective line  $\mathbb{X}$  of the wild type. It is shown that any exceptional vector bundle  $M$  over  $\mathbb{X}$ , of rank at least two, can be realized as the middle component of a non-split short exact sequence of the following form

$$(\star) \quad 0 \longrightarrow Y^{\oplus v} \longrightarrow M \longrightarrow X^{\oplus u} \longrightarrow 0,$$

where the pair  $(X, Y)$  is an exceptional orthogonal pair, the ranks of the bundles  $X$  and  $Y$  are less than the rank of  $M$  and  $[u, v]$  is the dimension vector of a unique representation for the generalized Kronecker quiver with  $n := \dim_k \text{Ext}_{\mathbb{X}}^1(X, Y)$  arrows. Then using mathematical induction and an alternative description of the extensions space, it has been found that "almost all" exceptional modules over wild canonical algebras can be described by matrices with entries from the relations of the canonical algebra. The "almost all" means that in the every  $\tau_{\mathbb{X}}$ -orbit of exceptional modules, from a certain place all the modules have the expected matrix representations.

In addition, in the dissertation, the representations for all two-rank modules were constructed using cokernels of morphisms between sums of line bundles.

### **Matrix factorizations**

Let  $\mathbb{X}$  be a weighted projective line with a weight triple  $(a, b, c)$ . Consider the hypersurface  $S = k[x, y, z]/\langle f \rangle$ , where  $f$  is the triangle singularity of the form  $x^a + y^b + z^c$ . Let  $\mathbb{L}$  be the Picard group of the category  $\text{coh}(\mathbb{X})$ . Then  $S$  is an  $\mathbb{L}$ -graded algebra and there is an equivalence of the category  $\text{vect}(\mathbb{X})$  of vector bundles over  $\mathbb{X}$  and the category  $\text{CM}^{\mathbb{L}}(S)$  of (maximal) Cohen-Macaulay  $\mathbb{L}$ -graded modules. It is known that the categories  $\text{vect}(\mathbb{X})$  and  $\text{CM}^{\mathbb{L}}(S)$  are Frobenius categories and their stable categories  $\underline{\text{vect}}(\mathbb{X})$  and  $\underline{\text{CM}}^{\mathbb{L}}(S)$  are triangle equivalent. In addition, the stable category of (maximal) Cohen-Macaulay modules  $\underline{\text{CM}}^{\mathbb{L}}(S)$  is equivalent to the singularity category  $\text{Sing}^{\mathbb{L}}(S)$  which was introduced and studied by R.O. Buchweitz in the no-graded case and D. Orlov in the graded case.

The second aim of the dissertation is to describe the matrix factorization for all indecomposable vector bundles, in the cases where  $f$  is the domestic type, i.e. the sequence of weights  $(a, b, c)$  is, up to permutation,  $(2, 2, n)$  for  $n \geq 2$  or  $(2, 3, 3)$  or  $(2, 3, 4)$  or  $(2, 3, 5)$ . In addition, we investigate matrix factorizations corresponding to indecomposable bundles of rank two in the general case  $(a, b, c)$  independently of the representation type.

Our work is related to, but in methods and results different from, the determination of matrix factorizations for the  $\mathbb{Z}$ -graded simple singularities by H. Kajiura, K. Saito, A. Takahashi. In particular, we obtain symmetric matrix factorizations whose entries are scalar multiples of monomials, with scalars taken from  $\{0, \pm 1\}$

For this purpose, we work in the category of vector bundles over a weighted projective line  $\mathbb{X}$  and use equivalence of the categories  $\text{vect}(\mathbb{X})$  and  $\text{CM}^{\mathbb{L}}(S)$ . The matrix factorization for a CM-module  $M$  (= vector bundle over  $\mathbb{X}$ ) can be obtained from the minimal projective resolution for  $M$  over the ring  $S$ . Hence, we first calculate projective covers for all indecomposable CM-modules (= vector bundles over  $\mathbb{X}$ ). Using these projective covers for a vector bundle  $E$  a matrix factorization  $(u_E, v_E)$  for  $E$  can be determined and the entries of the matrices  $u_E$  and  $v_E$  are shown to be scalar multiples of  $\mathbb{L}$ -homogeneous monomials in  $T = k[x, y, z]$ . In order to describe the matrix factorization  $(u_E, v_E)$  we have to find these scalars. First, we put zeros as coefficients in all places where the Hom spaces vanish. Next scalars for the remaining coefficients are calculated, in particular we show that they can be chosen as 1 and  $-1$ . Finally we have to show that the calculated matrix factorization in fact represents particular module, for this we prove the fact that a vector bundle is, up to isomorphism, uniquely determined by its projective cover.

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